Symmetries and reduced systems of equations for three-boson and four-boson interactions

2002 J. Phys. A: Math. Gen. 35 8587

(http://iopscience.iop.org/0305-4470/35/40/316)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 8587–8602 PII: S0305-4470(02)31889-4

Symmetries and reduced systems of equations for three-boson and four-boson interactions

V A Andreev[1] **and O A Ivanova**[2]

[1] P N Lebedev Physical Institute, Russian Academy of Sciences, Leninskii Pr. 53, Moscow 117924, Russia
[2] Department of Mechanics and Mathematics, Moscow State University, 119899, Moscow, Russian Federation

E-mail: andr@sci.lebedev.ru

Received 10 December 2001, in final form 29 January 2002
Published 24 September 2002
Online at stacks.iop.org/JPhysA/35/8587

Abstract

A reduced system of equations is found for three-boson and four-boson interactions. The invariants and surfaces in the space of parameters connected with these systems are constructed. It is shown that at these surfaces the eigenvectors can be presented in power form. The connection between the ansatz and polynomial extensions of Lie algebras is briefly discussed. The computational procedures corresponding to the original equations and the method of polynomial algebras are compared.

PACS numbers: 03.65.−w, 42.50.−p

1. Introduction

The Bethe ansatz plays an important role in the theory of integrable quantum models [1–4]. For the first time it has been applied to lattice models such as the isotropic and anisotropic spin chain [1, 2] and the Hubbard model [5, 6]. It was also applied to the quantum versions of the sine–Gordon [7, 8] and nonlinear Schrödinger equations [9].

The other class of models was introduced by Gaudin [10]. The possibility of functional ansatz for them was shown by Sklyamin [11]. These models have been investigated in many works [12–15]. The algebraic ansatz was applied by Jurĉo to the Gaudin models related to nonlinear quantum optics [16]. They describe such processes as second-harmonic generation, three-boson interaction, the Dike model [17] and some cases of four-boson interaction.

The dynamics of three-boson interaction was investigated in our work [18]. It turned out that in this case the ansatz formalism has some specific properties distinguishing the three-boson interaction from other models. The eigenvectors of the Hamiltonian in this approach have the form

$$|\lambda_1, \ldots, \lambda_N\rangle = (N!)^{-1/2} B(\lambda_1) \cdots B(\lambda_N)|0\rangle. \tag{1}$$

Here $|0\rangle$ is a pseudovacuum vector and $B(\lambda_i)$ are some specific operators depending on the parameters λ_i. In most models vectors (1), corresponding to different sets $\{\lambda_1, \ldots, \lambda_N\}$, $\{\lambda'_1, \ldots, \lambda'_N\}$, are linearly independent. This condition together with the demand of (1) to be the eigenvector for the Hamiltonian gives us the equations for the parameters $\{\lambda_i, i = 1, \ldots, N\}$. And, in fact, in the spin chain models, in quantum sine–Gordon and nonlinear Schrödinger equations these vectors are linearly independent [19–21], but in the case of three-boson and four-boson interaction this is not so. In this model eigenvectors (1) with different $\{\lambda_i, i = 1, \ldots, N\}$ can be linearly dependent or even coincide.

Therefore the admissible parameters defining some eigenvector (1) occupy some surface S in space C^N. Equations of these surfaces have been found in [22]. In this work we shall show how the existence of such dependences can change the form of equations for the ansatz parameters. In more exact terms the structure of equations for the ansatz parameters is found to be more closely connected with the algebraic structure of the model. The Hamiltonians of Gaudin's models have the form of the sum of products of operators of two algebras. The original equations are connected only with the form of these algebras. However, it turns out that if one considers the concrete representations of these algebras, the structure of the corresponding equations can be simplified. We shall call these reduced equations.

We shall define the coordinate system at the parametric surfaces S and with its help compare the computational procedures corresponding to the original equations, the reduced Bethe equations and the method of polynomial algebras [23, 24]. The algebraic sense of the parameters λ_i will also be clarified.

2. Algebraic Bethe ansatz scheme

According to the usual Bethe ansatz scheme in terms of Gaudin models [16] one must consider the 2×2 matrix $L(\lambda)$ with operator matrix elements

$$L(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{2}$$

This matrix must be a solution of the classical Yang–Baxter equation

$$[L(\lambda) \otimes I, I \otimes L(\mu)] + [r(\lambda - \mu), L(\lambda) \otimes I + I \otimes L(\mu)] = 0. \tag{3}$$

Here $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $r(\lambda)$ is a 4×4 matrix

$$r(\lambda) = \begin{pmatrix} f(\lambda) & 0 & 0 & 0 \\ 0 & 0 & g(\lambda) & 0 \\ 0 & g(\lambda) & 0 & 0 \\ 0 & 0 & 0 & f(\lambda) \end{pmatrix}. \tag{4}$$

It follows from equation (3) that elements of matrix (2) must satisfy the following commutation relations:

$$\begin{aligned}
[A(\lambda), B(\mu)] &= g(\lambda - \mu)B(\lambda) - f(\lambda - \mu)B(\mu), \\
[A(\lambda), C(\mu)] &= f(\lambda - \mu)C(\mu) - g(\lambda - \mu)C(\lambda), \\
[D(\lambda), B(\mu)] &= f(\lambda - \mu)B(\mu) - g(\lambda - \mu)B(\lambda), \\
[D(\lambda), C(\mu)] &= g(\lambda - \mu)C(\lambda) - f(\lambda - \mu)C(\mu), \\
[B(\lambda), C(\mu)] &= g(\lambda - \mu)(A(\lambda) - A(\mu) - D(\lambda) + D(\mu)), \\
[A(\lambda), A(\mu)] &= [B(\lambda), B(\mu)] = [C(\lambda), C(\mu)] = [D(\lambda), D(\mu)] \\
&= [A(\lambda), D(\mu)] = 0.
\end{aligned} \tag{5}$$

The Hamiltonian of a three-boson interaction has the form

$$H = H_0 + gH_I = \omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2 + \omega_3 a_3^+ a_3 + g(a_1^+ a_2 a_3 + a_3^+ a_2^+ a_1), \qquad \omega_1 = \omega_2 + \omega_3. \tag{6}$$

In this case the elements of matrix $L(\lambda)$ can be taken as

$$A(\lambda) = -\frac{1}{2}\lambda - \frac{1}{\lambda}S_3, \qquad B(\lambda) = a_3^+ + \frac{1}{\lambda}S_+, \qquad C(\lambda) = a_3 + \frac{1}{\lambda}S_-, \qquad D(\lambda) = \frac{1}{2}\lambda + \frac{1}{\lambda}S_3 = -A(\lambda). \tag{7}$$

Here the two-boson realization of $SU(2)$ algebra is used:

$$S_+ = a_1^+ a_2, \qquad S_- = a_2^+ a_1, \qquad S_3 = \tfrac{1}{2}(a_1^+ a_1 - a_2^+ a_2). \tag{8}$$

It is easy to verify that if

$$f(\lambda) = g(\lambda) = \frac{1}{\lambda} \tag{9}$$

operators (7) satisfy commutation relations (5). The eigenfunctions of Hamiltonian (6) can be constructed in form (1) with the pseudovacuum vectors

$$|0\rangle_M = |0, M, 0\rangle, \qquad M = 0, 1, 2, \ldots. \tag{10}$$

These vectors satisfy the equations

$$C(\lambda)|0\rangle_M = 0, \qquad A(\lambda)|0\rangle_M = a(\lambda)|0\rangle_M, \qquad D(\lambda)|0\rangle_M = d(\lambda)|0\rangle_M \qquad a(\lambda) = -d(\lambda) = \frac{1}{2\lambda}(M - \lambda^2). \tag{11}$$

Now we shall find the equations for the parameters $\{\lambda_i, i = 1, \ldots, N\}$. With this aim one must consider the trace of the matrix $L^2(\lambda)$.

$$T(\lambda) = \tfrac{1}{2}\operatorname{Tr} L^2(\lambda). \tag{12}$$

This operator has the important property

$$[T(\lambda), T(\mu)] = 0, \tag{13}$$

$$[T(\lambda), B(\mu)] = g(\lambda - \mu)(A(\mu)B(\lambda) - B(\lambda)D(\mu)) + f(\lambda - \mu)(B(\mu)D(\mu) - A(\lambda)B(\mu)). \tag{14}$$

The pseudovacuum vector (10) is the eigenvector for operator (12)

$$T(\lambda)|0\rangle_M = t_{M,0}(\lambda)|0\rangle_M, \qquad t_{M,0}(\lambda) = \tfrac{1}{2}(a^2(\lambda) + d^2(\lambda) - a'(\lambda) + d'(\lambda)). \tag{15}$$

In the case of three-boson interaction, i.e. for (7), (8) and (11), we have

$$T(\lambda) = \frac{1}{4}\lambda^2 + \left(a_3^+ a_3 + \frac{1}{2}\right) + S_3 + \frac{1}{\lambda^2}\left(S_3^2 + \frac{1}{2}(S_+S_- + S_-S_+)\right) + \frac{1}{\lambda}(a_3 S_+ + a_3^+ S_-), \qquad t_{M,0}(\lambda) = \frac{1}{4}\lambda^2 - \frac{M}{2} + \frac{1}{2} + \frac{1}{\lambda^2}\frac{M}{2}\left(\frac{M}{2} + 1\right). \tag{16}$$

It is obvious that the operator $T(\lambda)$ commutes both with the whole Hamiltonian (6) and with its interacting part H_I:

$$[T(\lambda), H] = [T(\lambda), H_I] = 0 \tag{17}$$

and

$$H_I = \operatorname{res} T(\lambda), \qquad \lambda \to 0. \tag{18}$$

We are looking for the eigenvectors of the operator $T(\lambda)$ in the form (1)

$$\Psi_{M,N}(\lambda_1,\ldots,\lambda_N) \equiv |\lambda_1,\ldots,\lambda_N\rangle_M = (N!)^{-\frac{1}{2}} B(\lambda_1)\cdots B(\lambda_N)|0\rangle_M \qquad N = 0, 1, 2, \ldots. \tag{19}$$

Acting on this by operator (12) and using the commutation relations (5) and (14) one can find the equation [16]

$$\begin{aligned} T(\lambda)|\lambda_1,\ldots,\lambda_N\rangle_M = {} & \left\{\sum_{i=1}^{N} f^2(\lambda-\lambda_i) - g^2(\lambda-\lambda_i) + \sum_{j\neq i} f(\lambda-\lambda_i)f(\lambda-\lambda_j) + t_{M,o}(\lambda)\right. \\ & \left. + \sum_{i=1}^{N} f(\lambda-\lambda_i)(d(\lambda)-a(\lambda))\right\}|\lambda_1,\ldots,\lambda_N\rangle_M \\ & + \sum_{i=1}^{N} g(\lambda-\lambda_i)\left\{a(\lambda_i)-d(\lambda_i)-2\sum_{j\neq i} f(\lambda_i-\lambda_j)\right\} \\ & \times |\lambda_1,\ldots,\lambda_{i-1},\lambda,\lambda_{i+1},\ldots,\lambda_N\rangle_M. \end{aligned} \tag{20}$$

As we want vector (19) to be the eigenvector of operator (12), it follows from this equation that the additional terms on the right-hand side of (20), containing vectors

$$\Psi^i_{M,N} = |\lambda_1,\ldots,\lambda_{i-1},\lambda_0,\lambda_{i+1},\ldots,\lambda_N\rangle, \tag{21}$$

must be equal to zero. If all these vectors are linearly independent and orthogonal to vector (19) we have the system

$$a(\lambda_i) - d(\lambda_i) - 2\sum_{j\neq i} f(\lambda_i-\lambda_j) = 0, \qquad i = 1,\ldots,N. \tag{22}$$

In the case of (9) and (11) this system takes the form

$$\lambda_i + \sum_{j\neq i}\frac{2}{\lambda_i-\lambda_j} = \frac{M}{\lambda_i}, \qquad i = 1,\ldots,N. \tag{23}$$

This very system (23) is usually regarded as the system of Bethe equations for the three-boson interaction (see for example [16]). It is clear from (20) that conditions (23) are necessary and sufficient for vector (19) to be the eigenvector of operator (16) only if all vectors (21) are linearly independent, but in the case of three-boson interaction these vectors are linearly independent only if $N \leqslant M$ and if $N > M$ vectors (19) and (21) are linearly dependent. As a result of this fact conditions (23) can be weakened and replaced by the another system. Now we shall find these equations.

Note that only the interaction part H_I of the Hamiltonian (6) can be presented in terms of the elements of matrix $L(\lambda)$ (2), but the selfenergy H_0 and interaction H_I parts commute.

$$[H_0, H_I] = 0. \tag{24}$$

Therefore the eigenvalue $E_{M,N}$ of the Hamiltonian H for the eigenvector $\Psi_{M,N}$ (19) is a sum [16]

$$E_{M,N} = E_0 + E_I = M\omega_2 + N\omega_3 + g\sum_{i=1}^{N}\lambda_i. \tag{25}$$

Here λ_i are solutions of system (23).

We see that the selfenergy part E_0 depends on the frequencies ω_2, ω_3 and on the integer numbers M, N that fix the state. The interaction part E_I depends on the integer numbers M, N and on the interaction constant g.

The energy of the ground state $\Psi_{0,0}$ is equal to zero: $E_{0,0} = 0$. The energies of the pseudovacuum states $\Psi_{M,0}$ are equal to their selfenergy parts: $E_{M,0} = M\omega_2$. The energy of interaction therefore appears only for excitations above the pseudovacuum states (10).

It was shown in [18] that the space D of the eigenvectors of Hamiltonian (6) is the direct sum of subspaces $D_{M,N}$

$$D = \bigoplus_{M,N=0}^{\infty} D_{M,N}, \qquad D_{M,N} = \{|K, M-K, N-K\rangle, K = 0, 1, \ldots, \min(M, N)\}, \tag{26}$$

$\dim D_{M,N} = \min(M, N) + 1$.

Let us consider some examples of eigenvectors of the Hamiltonian (6).

(0) $M = 0$. The pseudovacuum vector is $\Psi_{0,0} = |0, 0, 0\rangle$. All spaces $D_{0,N}$, $N \geqslant 0$ are one dimensional and contain the basis vectors $\Psi_{0,N} = |0, 0, N\rangle$. Their eigenvalues are $E_{0,N} = N\omega_3$.

(1) $M = 1$. The pseudovacuum vector is $\Psi_{1,0} = |0, 1, 0\rangle$. The space $D_{1,0}$ is one dimensional and contains the vector $\Psi_{1,0}$. It can be easily verified that there are only two linearly independent vectors among the set

$$\{B(\lambda_1^i) \cdots B(\lambda_N^i)|0, 1, 0\rangle, i = 1, 2, 3, \ldots\}. \tag{27}$$

Here $\{(\lambda_1^i, \ldots, \lambda_N^i), i = 1, 2, 3, \ldots\}$ are arbitrary sets of N numbers. Therefore all spaces $D_{1,N}$, $N \geqslant 1$ are two dimensional. They contain the basis vectors

$$\Psi_{1,N}^{\pm} = |0, 1, N\rangle \pm |1, 0, N-1\rangle \tag{28}$$

with eigenvalues $E_{1,N}^{\pm} = \omega_2 + N\omega_3 \pm g\sqrt{N}$.

(2) $M = 2$. The pseudovacuum vector is $\Psi_{2,0} = |0, 2, 0\rangle$. The space $D_{2,0}$ is one dimensional and contains the vector $\Psi_{2,0}$. The space $D_{2,1}$ is two dimensional and its basis vectors are

$$\Psi_{2,N}^{\pm} = |0, 2, 1\rangle \pm |1, 1, 0\rangle \tag{29}$$

with eigenvalues $E_{2,1}^{\pm} = 2\omega_2 + \omega_3 \pm g\sqrt{2}$. All other spaces $D_{1,N}$, $N \geqslant 2$ are three dimensional. They contain the basis vectors

$$\Psi_{2,N}^{\pm} = |0, 2, N\rangle + \sqrt{(N-1)/N}|2, 0, N-2\rangle \pm \sqrt{(2N-1)/N}|1, 1, N-1\rangle \tag{30}$$

with eigenvalues $E_{2,N}^{\pm} = 2\omega_2 + N\omega_3 \pm g\sqrt{4N-2}$, and the vector

$$\Psi_{2,N}^{0} = |0, 2, N\rangle - \sqrt{N/(N-1)}|2, 0, N-2\rangle, \tag{31}$$

whose energy does not depend on the interaction: $E_{2,N}^{0} = 2\omega_2 + N\omega_3$.

Some other examples can be found in [18].

It was shown in [25] that in the case of the Dicke model the conditions $N < M$ and $N > M$ correspond to the cases of the weak- and strong-field regimes.

In this work we discuss only solutions of the stationary Schrödinger equation with the Hamiltonians of three- and four-boson interaction. The time dynamics of these models was discussed in [26, 27]. In those works the evolution of Wigner and Husimi functions was investigated.

3. Reduced system of Bethe equations for three-boson interaction

The explicit form of vector (19) was found for the three-boson interaction in [18].

$$\Psi_{M,N}(\lambda_1, \ldots, \lambda_N) = \sum_{k=0}^{\min(N,M)} \left(\frac{k!M!(N-k)!}{(M-k)!N!}\right)^{\frac{1}{2}} \left(\sum_{i_1<\cdots<i_k} (\lambda_{i_1} \cdots \lambda_{i_k})^{-1}\right)|k, M-k, N-k\rangle. \tag{32}$$

We see that for every $N > M$ this sum contains $(M + 1)$ members, therefore among $(N + 1)$ vectors (19), (21) only $(M + 1)$ are linearly independent. Let us construct the set of linearly independent vectors. As the first member of this set we shall take vector (32); the other M vectors have the form

$$\Psi_l \equiv |0, M, N\rangle - \left(\frac{(M-l)!N!}{l!M!(N-l)!}\right)^{\frac{1}{2}} \left(\sum_{i_1<\cdots<i_l} (\lambda_{i_1}\ldots\lambda_{i_l})^{-1}\right)^{-1} |l, M-l, N-l\rangle. \tag{33}$$

$l = 1, \ldots, M$.

All vectors (33) are linearly independent and orthogonal to the vector (32). Let us calculate the scalar product

$$c_l^j = \langle \Psi_l \mid \Psi_{M,N}^j \rangle = 1 - \left(\sum_{i_1<\cdots<i_l}^{j} (\lambda_{i_1}\ldots\lambda_{i_l})^{-1}\right)\left(\sum_{i_1<\cdots<i_l} (\lambda_{i_1}\ldots\lambda_{i_l})^{-1}\right)^{-1}. \tag{34}$$

In the sum $\sum^J$ parameter λ_j is replaced by parameter λ.

Substituting in (20) vectors (33) instead of vectors (21) one obtains the system of equations

$$\sum_{i=1}^{N} g(\lambda-\lambda_i)\left\{a(\lambda_i) - d(\lambda_i) - 2\sum_{j\neq i} f(\lambda_i-\lambda_j)\right\}c_l^i = 0, \qquad l = 1, \ldots, M. \tag{35}$$

Taking into account formulae (9) and (11), system (35) can be transformed into the system

$$\sum_{i=1}^{N} (\lambda_i)^{-l}\left(\lambda_i - \frac{M}{\lambda_i} + 2\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j}\right) = 0, \qquad l = 1, \ldots, M. \tag{36}$$

Thus, if $M < N$ one can use instead of the usual system (23) the reduced system (28). System (23) contains N equations and N unknown parameters λ_i. Its solutions are points in N-dimensional complex space. System (36) contains M equations for N unknown parameters; its solutions are $(N-M)$-dimensional complex surfaces in an N-dimensional space of parameters. Each surface consists of M connected components. All points from the same connected component give the same vector (32), so

$$\Psi_{M,N}(\lambda_1, \ldots, \lambda_N) \equiv \Psi_{M,N}(\mu_1, \ldots, \mu_N),$$

if $\{\lambda_i\}$ and $\{\mu_i\}$ belong to the same connected component. Equations (23) give us the specific points from each component. For $M = 1$, $N = 1, 2, \ldots, 14$ these points have been evaluated in [18]. Thus we see that the usual Bethe equations (23) give the correct set of parameters, but this set is not unique. Note that equation (36), in contrast to (23), permit solutions containing equal $\lambda_i = \lambda_j$.

4. Spectrum formula and invariants of the surfaces

The eigenvalue of the operator $T(\lambda)$ was found in [16].

$$t_{M,N}(\lambda) = \sum_{i=1}^{N}\Big[f^2(\lambda-\lambda_i) - g^2(\lambda-\lambda_i) + f(\lambda-\lambda_i)(d(\lambda)-a(\lambda)) + \sum_{i\neq j} f(\lambda-\lambda_i)f(\lambda-\lambda_j)\Big] + t_{M,0}(\lambda). \tag{37}$$

In the case of (9), (11) and (15) it takes the form

$$t_{M,N}(\lambda) = \sum_{i=1}^{N}\left\{\frac{\lambda^2-M}{\lambda(\lambda-\lambda_i)} + \sum_{j\neq i}\frac{1}{(\lambda-\lambda_i)(\lambda-\lambda_j)}\right\} + \frac{1}{4}\lambda^2 - \frac{M}{2} + \frac{1}{2} + \frac{1}{\lambda^2}\frac{M}{2}\left(\frac{M}{2}+1\right). \tag{38}$$

This formula is valid if $N \leqslant M$, but if $N > M$ it is valid only for those $\{\lambda_i, i = 1, \ldots, N\}$ which satisfy equations (23); however, if parameters $\{\lambda_i, i = 1, \ldots, N\}$ satisfy only the weaker system (36) formula (38) is not valid. We must add to it some additional terms, connected with the fact that now vectors $\Psi_{M,N}$ and $\Psi^i_{M,N}$ are linearly dependent. Such an addition $\Delta_{M,N}$ to eigenvalue (38) has the form

$$\Delta_{M,N}(\lambda) = C^{-2}_{M,N} \sum_{i=1}^{N} \langle \Psi^i_{M,N} | \Psi_{M,N} \rangle F_i(\lambda). \tag{39}$$

Here

$$C^2_{M,N} = \sum_{k=0}^{M} \left(\frac{k!(N-k)!M!}{(M-k)!N!} \right) \left(\sum_{i_1 < \cdots < i_k} (\lambda_{i_1} \ldots \lambda_{i_k})^{-1} \right)^2$$

is the square of the norm of vector (23) and

$$F_i(\lambda) = \frac{1}{\lambda - \lambda_i} \left(\frac{M}{\lambda_i} - \lambda_i - 2 \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \right).$$

Using the fact

$$\langle \Psi^i_{M,N} | \Psi_{M,N} \rangle = \sum_{k=0}^{M} \left(\frac{k!(N-k)!M!}{(M-k)!N!} \right) \left(\sum_{i_1 < \cdots < i_k}^{i} (\lambda_{i_1} \ldots \lambda_{i_k})^{-1} \right) \left(\sum_{i_1 < \cdots < i_k} (\lambda_{i_1} \ldots \lambda_{i_k})^{-1} \right)$$

we obtain

$$\Delta_{M,N}(\lambda) = \sum_{i=1}^{N} F_i(\lambda). \tag{40}$$

Therefore for an arbitrary set of parameters $\{\lambda_i, i = 1, \ldots, N\}$, satisfing system (36), the eigenvalue of the operator $T(\lambda)$ has the form

$$\begin{aligned} t_{M,N}(\lambda) = \sum_{i=1}^{N} \Bigg\{ & \frac{\lambda^2 - M}{\lambda(\lambda - \lambda_i)} + \sum_{j \neq i} \frac{1}{(\lambda - \lambda_i)(\lambda - \lambda_j)} + \frac{M - \lambda_i^2}{\lambda_i(\lambda - \lambda_i)} \\ & - 2 \sum_{j \neq i} \frac{1}{(\lambda - \lambda_i)(\lambda_i - \lambda_j)} \Bigg\} + \frac{1}{4}\lambda^2 - \frac{M}{2} + \frac{1}{2} + \frac{1}{\lambda^2} \frac{M}{2} \left(\frac{M}{2} + 1 \right) \\ = & \frac{1}{\lambda} \sum_{i=1}^{N} \frac{M}{\lambda_i} + N + \frac{1}{4}\lambda^2 - \frac{M}{2} + \frac{1}{2} + \frac{1}{\lambda^2} \frac{M}{2} \left(\frac{M}{2} + 1 \right). \end{aligned} \tag{41}$$

Comparing formulae (16) and (41) we have

$$E_I = \sum_{i=1}^{N} \frac{M}{\lambda_i}, \qquad H_I \Psi_{M,N} = E_I \Psi_{M,N}. \tag{42}$$

For E_I the following expression was found in [16]:

$$E_I = \sum_{i=1}^{N} \lambda_i. \tag{43}$$

However, formula (43) is valid only for those $\{\lambda_i\}$ which satisfy equations (23), while formula (41) is valid for all sets $\{\lambda_i\}$, belonging to surfaces (36). So we see that the quantity

$$S_1 = \sum_{i=1}^{N} \lambda_i^{-1} = \frac{1}{M} E_I$$

is an invariant of the surface, to be more exact of its connected component. The other invariants have the form

$$S_k = \sum_{i_1<\cdots<i_k} (\lambda_{i_1} \dots \lambda_{i_k})^{-1} = (-1)^k \frac{N!}{(N-k)!} \frac{A_{M-k}}{A_M}, \qquad k = 1, \dots, M. \tag{44}$$

Here A_l is an $l \times l$-lower principal minor of the matrix A_{MN}

$$A_{MN} = \| H_I - EI \| = \| h_{ij} - E\delta_{ij} \| = \| a_{ij} \|, \qquad i, j = 1, \dots, M+1. \tag{45}$$

$$\begin{aligned} &h_{ij} = \delta_{i+1,j} h_i + \delta_{i,j+1} h_j, \qquad h_i = ((M+1-i)(N+1-i))^{1/2}. \\ &A_l = \| a^l_{ij} \|, \qquad i, j = 1, \dots, l, \quad a^l_{ij} = a_{M+1-l+i, M+1-l+j}. \end{aligned} \tag{46}$$

Equation $\det A_{MN} = 0$ gives us the eigenvalues of vectors (32). Formulae (44) can be found by comparing formula (32) with the usual structure of eigenvectors of the matrix (45).

It was shown in [18] that the space D of the eigenvectors of Hamiltonian (6) is the direct sum of subspaces $D_{M,N}$:

$$\begin{aligned} &D = \bigoplus_{M,N=0}^{\infty} D_{M,N}, \qquad D_{M,N} = \{|K, M-K, N-K\rangle, K = 0, 1, \dots \min(M, N)\}, \\ &\dim D_{M,N} = \min(M, N) + 1. \end{aligned} \tag{47}$$

The vectors from these subspaces are the eigenvectors of two operators

$$K_1 = S_3^2 + \tfrac{1}{2}(S_+S_- + S_-S_+), \qquad K_2 = (a_3^+ a_3 + \tfrac{1}{2}) + S_3 \tag{48}$$

with the following eigenstates (41):

$$K_1|_{D_{M,N}} = \frac{M}{2}\left(\frac{M}{2} + 1\right), \qquad K_2|_{D_{M,N}} = N - \frac{M}{2} + \frac{1}{2}. \tag{49}$$

In the case $N \geqslant M$ $D_{M,N}$ is the space of $(M+1)$-dimensional representation of the algebra $SU(2)$. If $N < M$ the space $D_{M,N}$ forms only part of the space of the representation. Different vectors (32) from the space $D_{M,N}$ have different energy $E = \sum_{i=1}^{N} M/\lambda_i$. The operators K_1, K_2, H_I are the coefficients attached to different degrees of the parameter λ in formula (16) for the operator $\mathrm{Tr}(\lambda)$ (λ^{-2}, λ^0, λ^{-1} correspondingly).

The original equations for the Bethe ansatz parameters (23) correspond to the case $N \leqslant M$, and $\dim S = 0$, therefore S is a set of points in the space C^N. The reduced Bethe equations (36) correspond to the case $N > M$ and $\dim S = N - M > 0$. The number N of equations of the original Bethe system (23) can be reduced now by the dimension of surface S

$$N - \dim S = M.$$

M is the number of equations of the reduced system (36). Hamiltonian H_I (6) can be presented by the generators of algebra $SU(2)\{S_3, S_+, S_-\}$ and Heisenberg's algebra $\{a_3, a_3^+\}$

$$H_I = a_3 S_- + a_3^+ S_+. \tag{50}$$

They are the same for all N, M. The structure of the reduced system (36) is more closely connected with concrete values of these numbers. In table 1 of [18], the spaces of the representations of algebra $SU(2)$ are given by the rows, the infinite columns correspond to the representation of Heisenberg's algebra $\{a_3, a_3^+\}$ and the spaces $D_{M,N}$ are the diagonals. The structure of these diagonals and of the reduced Bethe equations is defined by the intersection of the spaces of representation of these two algebras.

5. Asymptotics and generating functions for the eigenvectors

The fact that for $N > M$ series (32) contains only $(M+1)$ members can be used for calculating the asymptotics of the vectors $\Psi_{M,N}$ for large N. With this aim let us consider the $(M+1)$-dimensional irreducible representation of algebra $SU(2)$. In the space of this representation one can find the eigenvectors of the operator $S_+ + S_-$

$$(S_+ + S_-)\Psi_M^m = E_M^m \Psi_M^m, \qquad m = 0, 1, \ldots, M. \tag{51}$$

The solutions of this equation have the form

$$\Psi_M^m = \sum_{k=0}^{M} C_{M,k}^m l_k, \qquad C_{M,0}^m = 1. \tag{52}$$

Here $\{l_k,\ k = 0, \ldots, M\}$ is a basis in the space of the representation, such that

$$S_3 l_k = \left(k - \frac{M}{2}\right) l_k.$$

Then it follows from (45) that vector (32)

$$\Psi_{M,N}^m \to \sum_{k=0}^{M} C_{M,k}^m |k, M-k, N\rangle \qquad \text{and} \qquad E_{M,n}^m \to \sqrt{N} E_M^m \qquad \text{if } N \to \infty. \tag{53}$$

As a basis $\{l_k,\ k = 0, \ldots, M\}$ we can use $l_k = |k, M-k, 0\rangle$, then

$$\Psi_M^m = \sum_{k=0}^{M} C_{M,k}^m |k, M-k, 0\rangle$$

and equation (49) takes the form

$$\Psi_{M,N}^m \to (N!)^{-1/2} (a_3^+)^N \Psi_M^m \qquad \text{if } N \to \infty. \tag{54}$$

This is the simplest example of application of the formalism developed in this article. The more complicated asymptotic was found in [25].

6. Coordinate systems on the parametric surfaces

It was found that the equations of the parametric surfaces S_{N-M} have the form [22]

$$\sum_{i_1 < \cdots < i_k} (\lambda_{i_1} \ldots \lambda_{i_k})^{-1} = I_k = (-1)^k \frac{N!}{(N-k)!} \frac{A_{M-k}}{A_M}, \qquad k = 1, \ldots, M. \tag{55}$$

In order to fix a point on the surface S_{N-M} one must define other $(N-M)$ equations

$$\sum_{i_1 < \cdots < i_{M+l}} (\lambda_{i_1} \ldots \lambda_{i_{M+l}})^{-1} = J_l, \qquad l = 1, \ldots, N-M. \tag{56}$$

These numbers J_l can be considered as the coordinates of point λ on the surface S_{N-M}. Thus each point $\lambda = \lambda_1, \ldots, \lambda_N$ in N-dimensional complex space C^N is defined by two sets of numbers $\{I_k, k = 1, \ldots, M\}$ and $\{J_l, l = 1, \ldots, N-M\}$. Members I_k of the first set are the values of invariants (55) of solutions (32); they define the surface S_{N-M}. Members J_l of the second set are not the invariants of the solutions and define the point on the surface. However, in practice it is more convenient to consider the quantities

$$C_n = \sum_{i_1 < \cdots < i_n} (-1)^n (\lambda_{i_1} \ldots \lambda_{i_n}) \qquad n = 1, \ldots, N. \tag{57}$$

If $N > M$ the quantities C_n are not the invariants of solutions (32), but with their help one can construct the polynomials

$$P_N = \sum_{i=0}^{N} x^{N-i} C_i, \qquad C_0 = 1 \tag{58}$$

whose roots are the parameters $\lambda_1 \dots \lambda_N$. The quantities C_n can be directly found via matrix elements of the Hamiltonian H.

It was shown in [22] that in the case of $N \leqslant M$

$$C_n = \frac{1}{n!} A_n, \qquad n = 1, \dots, N, \tag{59}$$

where A_n is the lower major n-dimensional minor of $(N+1) \times (N+1)$ matrix A_{MN}.

If $N > M$ one must consider the extended $(N+1) \times (N+1)$ matrix $\tilde{A}_{MN}$ instead of the $(M+1) \times (M+1)$ matrix A_{MN} (45). This matrix has a block structure

$$\tilde{A}_{MN} = \begin{Vmatrix} A_{MN} & 0 \\ 0 & G_{MN} \end{Vmatrix}. \tag{60}$$

The upper block coincides with the matrix A_{MN}. The lower $(N-M) \times (N-M)$-dimensional block G_{MN} has such a structure that numerical values of all quantities C_n can be calculated with the help of the formula

$$C_n = \frac{1}{n!} \tilde{A}_n, \tag{61}$$

where $\tilde{A}_n$ is the lower major n-dimensional minor of the $(N+1) \times (N+1)$ extended matrix $\tilde{A}_{MN}$ (60). Since matrix $\tilde{A}_{MN}$ has a block structure the minor $\tilde{A}_n$ can be factorized for all $n \geqslant N - M$:

$$\tilde{A}_n = G \cdot A_{n-N+M}, \qquad G = \det \|G_{MN}\|. \tag{62}$$

Now we can find the connection between I_K, J_l and minors of the matrix $\tilde{A}_{MN}$.

$$I_K = \frac{C_{N-K}}{C_N} = (-1)^K \frac{N!}{(N-K)!} \frac{\tilde{A}_{N-K}}{\tilde{A}_N} = (-1)^K \frac{N!}{(N-K)!} \frac{A_{M-K}}{A_M} \qquad K = 1, \dots, M \tag{63}$$

$$J_l = \frac{C_{N-M-l}}{C_N} = (-1)^{M+l} \frac{N!}{(N-M-l)!} \frac{\tilde{A}_{N-M-l}}{\tilde{A}_N}, \qquad l = 1, \dots, N-M. \tag{64}$$

We see that the invariants I_K of solution (32) are the functions of the elements of the matrix A_{MN} only, which has direct physical sense. The coordinates J_l of the points on the parametric surface depend on the elements of the lower unphysical block G_{MN} of matrix $\tilde{A}_{MN}$. More precisely quantities J_l are functions of the lower major minors of the matrix G_{MN}. There are $N - M$ such minors, including the determinant G,

$$A_{J,1}, \dots, A_{J,N-M} = G. \tag{65}$$

These $(N - M)$ numbers can be also considered as the coordinates of the point on the surface S_{N-M}. Defining any explicit form for the elements of the matrix G_{MN}, one can use them as the third set of coordinates on the surface S_{N-M}.

For example

$$G_{MN} = \|d_{ij} - E\delta_{ij}\|, \qquad d_{ij} = (\delta_{i,j+1} + \delta_{i+1,j}) r_i + \delta_{ij} d. \tag{66}$$

Now parameters $\{d, r_i, i = 1, \dots, N - M - 1\}$ can be considered as the coordinates of the point on the surface S_{N-M}.

It is necessary to note that the invariants I_K are not all independent. Each solution (32) is defined by three parameters M, N, E_I. Correspondingly among the invariants I_K only three are independent; all others can be found with the help of the recurrence condition

$$(N-K+1)I_{K-1}+E_I I_K+\frac{h_{M-K+1}}{N-K}I_{K+1}=0. \tag{67}$$

As we know there is one point marked out on the surface S_{N-M}. This point is defined by the original system (23). Let us find its coordinates in system (67). This can be done by comparing the $SU(2)$ and $SU(1,1)$ forms of the Bethe ansatz for three-boson interaction, or, more exactly, formulae (32) and (45) with formulae (A.2) and (A.4) in [22]. Then it is easy to see that

$$d=0, \qquad r_l=((N-M-l)(-l)(M+1+l))^{1/2}, \qquad l=1,\dots,N-M-1 \tag{68}$$

are the coordinates of this point.

7. The comparison of different computational procedures

Let us compare the computational procedures corresponding to the original and reduced systems for the Bethe ansatz parameters. In the case of system (23) one must deal with the extended $(N+1)\times(N+1)$ matrix $\tilde{A}_{MN}$ (60), which contains physical and unphysical blocks A_{MN} and G_{MN} correspondingly. First one must solve the equation

$$t|A_{MN}|=0 \tag{69}$$

and find the eigenvalues E_I of the Hamiltonian H_I. Then it is necessary to evaluate the minors $\tilde{A}_n$ of the matrix $\tilde{A}_{MN}$ and construct a polynomial

$$P_N=\sum_{n=0}^{N}\frac{x^{N-n}}{n!}\tilde{A}_n. \tag{70}$$

The roots of this polynomial are the desired quantities r_n.

In the case of the reduced system (36) we must first evaluate the invariants I_K with the help of matrix A_{MN} and formula (55). Then we must divide N parameters λ_n into two parts

$$\begin{aligned}&\lambda_i^{-1}=\mu_i, \qquad i=1,\dots,M\\ &\lambda_{M+j}^{-1}=\nu_j, \qquad j=1,\dots,N-M.\end{aligned} \tag{71}$$

The values of ν_j can be taken arbitrarily. We shall fix them and evaluate the quantities

$$Q_l=\sum_{j_1<\dots<j_l}(\nu_{j1}\dots\nu_{jl}), \qquad l=1,\dots,N-M. \tag{72}$$

The parameters μ_i now satisfy the equations

$$\sum_{i_1<\dots<i_K}(\mu_{i_1}\dots\mu_{i_K})=I_K-Q_K-\sum_{l=1}^{K-1}I_{K-l}Q_l=L_K, \qquad K=1,\dots,M. \tag{73}$$

Therefore M parameters μ_i are the roots of the polynomial

$$\tilde{P}_M=\sum_{K=0}^{M}x^K L_{M-K}. \tag{74}$$

In the case of the original Bethe equation (23) one must therefore solve first equation (69), of order $(M+1)$, and then solve the equation $P_N=0$, of order N; in the case of the reduced Bethe equations one must solve first the same equation (69), and then the equation $\tilde{P}_M=0$, whose order is M. If $M\ll N$ the difference in computations may be very large.

In the next section we consider another approach to the same problem, based on the infinite-dimensional polynomial algebras. In this method it is enough to solve only equation (69) to find eigenvalues and eigenstates of the Hamiltonian H. Therefore the knowledge of the Bethe parameters λ is necessary not for constructing the eigenstates of the Hamiltonian H_I, but for investigating the algebraic structure of the model. The knowledge of the parameters λ helps us to represent some operators from polynomial algebra $SU_d^{(1,2)}(2)$ in terms of $SU(2)$ and Heisenberg's algebras.

8. Reduced system of Bethe equations for four-boson interaction

Let us consider now the four-boson interaction. There exist two different variants of this model described by the Hamiltonians

$$H_{I1} = a_1 a_2^+ a_3^+ a_4^+ + a_1^+ a_2 a_3 a_4, \qquad \omega_1 = \omega_2 + \omega_3 + \omega_4 \tag{75}$$

$$H_{I2} = a_1 a_2^+ a_3 a_4^+ + a_1^+ a_2 a_3^+ a_4, \qquad \omega_1 + \omega_3 = \omega_2 + \omega_4. \tag{76}$$

Hamiltonian (75) describes the process of fusion of three particles into one particle and decay of one particle into three particles. Hamiltonian (76) describes the process of transformation of two particles into two particles with different energies. The spaces of the eigenvectors of Hamiltonians (75) and (76) have different structure.

Let us consider the system described by Hamiltonian (75). The general scheme of the Bethe ansatz method is the same as in the case of three-boson interaction. The eigenfunctions of Hamiltonian (75) can be constructed with the help of the operator

$$B(\lambda) = \frac{1}{\lambda} a_1^+ a_2 + a_3^+ a_4^+. \tag{77}$$

There exist two types of pseudovacuum vector

$$|0\rangle_{n_2,n_3} = |0, n_2, n_3, 0\rangle, \tag{78}$$

$$|0\rangle_{n_2,n_4} = |0, n_2, 0, n_4\rangle. \tag{79}$$

The eigenfunctions of Hamiltonian (75) constructed with the help of the Bethe ansatz method have the form

$$\begin{aligned} B(\lambda_1)\cdots B(\lambda_N)|0\rangle_{n_2,n_3} &= \sum_{k=0}^{\min(N,n_2)} I_k \left(\frac{n_2!k!(n_3+N-k)!(N-k)!}{(n_2-k)!n_3!} \right)^{1/2} \\ &\quad \times |k, n_2-k, n_3+N-k, N-k\rangle, \\ B(\lambda_1)\cdots B(\lambda_N)|0\rangle_{n_2,n_4} &= \sum_{k=0}^{\min(N,n_2)} I_k \left(\frac{n_2!k!(n_4+N-k)!(N-k)!}{(n_2-k)!n_4!} \right)^{1/2} \\ &\quad \times |k, n_2-k, N-k, n_4+N-k\rangle, \end{aligned} \tag{80}$$

where I_k is defined by the formula (55).

In the case of $N \leqslant n_2$ the vectors $|\lambda_1, \ldots, \lambda_N\rangle$ are linearly independent and parameters $\lambda_1, \ldots, \lambda_N$ must satisfy the usual system of Bethe equations

$$\frac{n_2}{\lambda_i} + (n_4+1)\lambda_i - 2\sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j} = 0, \qquad i = 1, \ldots, N. \tag{81}$$

If $N > n_2$ one can use instead of system (81) the reduced system of Bethe equations

$$\sum_{i=1}^{N} (\lambda_i)^{-l} \left(\frac{n_2}{\lambda_i} + (n_4+1)\lambda_i - 2\sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j} \right) = 0 \qquad l = 1, \ldots, n_2. \tag{82}$$

We see that the structure of systems (81) and (82) coincides with the structure of the analogous systems for three-boson interaction.

Let us consider now the system described by Hamiltonian (76). The eigenfunctions of this Hamiltonian can be constructed with the help of the operator

$$B(\lambda) = \frac{1}{\lambda} a_1^+ a_2 + a_3^+ a_4 \tag{83}$$

and pseudovacuum vectors

$$|0\rangle_{n_2,n_4} = |0, n_2, 0, n_4\rangle. \tag{84}$$

The form of these eigenfunctions depends on n_2, n_4, N. Let $n_2 \leqslant n_4$, then the following cases must be considered.

(i) $N \leqslant n_2 < n_4$

$$\Psi_N^{n_2,n_4}(\lambda_1, \ldots, \lambda_N) = \sum_{k=0}^{N} I_k \left(\frac{n_2!k!(N-k)!n_4!}{(n_2-k)!(n_4-N+k)!} \right)^{1/2} \times |k, n_2-k, N-k, n_4-N+k\rangle. \tag{85}$$

(ii) $n_2 \leqslant N \leqslant n_4$

$$\Psi_N^{n_2,n_4}(\lambda_1, \ldots, \lambda_N) = \sum_{k=0}^{n_2} I_k \left(\frac{n_2!k!(N-k)!n_4!}{(n_2-k)!(n_4-N+k)!} \right)^{1/2} \times |k, n_2-k, N-k, n_4-N+k\rangle. \tag{86}$$

(iii) $n_2 \leqslant n_4 \leqslant N \leqslant n_2 + n_4$

$$\Psi_N^{n_2,n_4}(\lambda_1, \ldots, \lambda_N) = \sum_{k=N-n_4}^{n_2} I_k \left(\frac{n_2!k!(N-k)!n_4!}{(n_2-k)!(n_4-N+k)!} \right)^{1/2} \times |k, n_2-k, N-k, n_4-N+k\rangle. \tag{87}$$

(iv) $n_2 + n_4 \leqslant N$

$$\Psi_N^{n_2,n_4}(\lambda_1, \ldots, \lambda_N) \equiv 0.$$

The usual system of Bethe equations for Hamiltonian (76) has the form

$$\frac{n_2}{\lambda_i} - n_4\lambda_i - 2\sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j} = 0, \qquad i = 1, \ldots, N. \tag{88}$$

In case (i) the reduced system does not exist and all parameters λ_i are defined by system (81).

In case (ii) the reduced system of Bethe equations has a form similar to the form of systems (81) and (82).

$$\sum_{i=1}^{N} (\lambda_i)^{-l} \left(\frac{n_2}{\lambda_i} - n_4\lambda_i - 2\sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j} \right) = 0 \qquad l = 1, \ldots, n_2. \tag{89}$$

And in case (iii) the reduced system of Bethe equations differs from systems (81) and (82). It has the form

$$\sum_{i=1}^{N} (\lambda_i)^{l} \left(\frac{n_2}{\lambda_i} - n_4\lambda_i - 2\sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j} \right) = 0 \qquad l = 0, \ldots, n_2 - 1. \tag{90}$$

9. Bethe ansatz and polynomial extensions of Lie algebras

The other approach to quantum optical problems based on the polynomial extensions of Lie algebras was put forth in [23, 24]. According to this the Hamiltonian of the system must be presented in the form

$$H = \sum_{i=1}^{N} \alpha_i Y_i, \qquad \alpha_i = \text{const}, \tag{91}$$

where Y_i are operators from the enveloping algebra of some finite-dimensional algebra G. These operators satisfy the commutation relations

$$[Y_i, Y_j] = f_{ij}(Y_k), \qquad i, j, k = 1, \ldots, N, \tag{92}$$

where $f_{i,j}$ are structural polynomials of finite order. Knowing these polynomials one can study the properties of Hamiltonian (91). In the case of three-boson interaction Hamiltonian (6) can be presented in the form

$$H = C + g(Y_+ + Y_-), \tag{93}$$

where

$$C = H_0 = \omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2 + \omega_3 a_3^+ a_3, \qquad Y_+ = a_1 a_2^+ a_3^+, \qquad Y_- = a_1^+ a_2 a_3. \tag{94}$$

These operators together with the additional operator

$$Y_0 = \tfrac{1}{3}(-a_1^+ a_1 + a_2^+ a_2 + a_3^+ a_3) \tag{95}$$

satisfy the commutation relations

$$[Y_0, Y_\pm] = \pm Y_\pm; \qquad [C, Y_{\pm 0}] = 0, \qquad [Y_-, Y_+] = \Psi_{1,2}(Y_0 + 1) - \Psi_{1,2}(Y_0). \tag{96}$$

The structural polynomial $\Psi_{1,2}(Y_0)$ defines the polynomial algebra $SU_d^{(1,2)}(2)$ with commutation relations (96). It has the form [23]

$$\begin{aligned} &\Psi_{1,2}(Y_0) = \tfrac{1}{4}(R_0 - Y_0 + 1)(2Y_0 + R_0 + R_1)(2Y_0 + R_0 - R_1), \\ &R_0 = \tfrac{1}{3}(2a_1^+ a_1 + a_2^+ a_2 + a_3^+ a_3), \qquad R_1 = -a_2^+ a_2 + a_3^+ a_3. \end{aligned} \tag{97}$$

The eigenvectors of Hamiltonian (93) can be sought as

$$|E\rangle = \sum_{n=1}^{s} Q_n(E)(Y_+)^n |s, k, 0\rangle. \tag{98}$$

The coefficients Q_n satisfy the recurrence relation

$$Q_{n-1} - E_I Q_n + \Psi_{1,2}(l_1 + n + 1) Q_{n+1} = 0. \tag{99}$$

Here E_I is an eigenvalue of the interaction Hamiltonian H_I and l_1 is an eigenvalue of operator Y_0

$$\begin{aligned} &H_I|E\rangle = (Y_+ + Y_-)|E\rangle = E_I|E\rangle, \\ &Y_0|s, k, 0\rangle = \tfrac{1}{3}(k - s)|s, k, 0\rangle = l_1|s, k, 0\rangle. \end{aligned}$$

Recurrence relations (99) are valid for all polynomial deformations $SU_d^{(p,q)}(2)$ of algebra $SU(2)$ with the specific polynomial $\Psi_{p,q}$ in each case. In the case of algebra (94)–(97) we have

$$\begin{aligned} &Q_{n-1} - E_I Q_n + (s - n)(n + 1)(n + 1 - k) Q_{n+1} = 0, \\ &Q_{-1} \equiv 0, \qquad Q_0 \equiv 1. \end{aligned} \tag{100}$$

The condition $Q_{s+1} = 0$ gives us the equation for the eigenvalues E_I of the operator H_I. This condition is equivalent to equation (69). We see that equation (69) and the recurrence relation

give us the coefficients Q_n and eigenstate (99) without using the Bethe parameters λ. These parameters are necessary for transforming the vector $|E\rangle$ from form (99) into form (1), or, more formally, for splitting the polynomial algebra $SU_q^{(1,2)}(p,q)$ into the composition of algebra $SU(2)$ and Heisenberg's algebra $\{a_3, a_3^+\}$.

By comparing formulae (32) and (99) for the eigenvectors of Hamiltonian (6) one can find the connection between polynomials Q_n and invariants S_i (44).

$$\begin{aligned} Q_i &= \left(\frac{(M-i)!(N-M+i)!}{M!(N-M)!i!}\right)^{1/2}\frac{S_{M-i}}{S_M}, \qquad i=0,\ldots,M,\ M<N \\ Q_i &= \left(\frac{(N-i)!i!(N-M)!}{N!(M-N+i)!}\right)^{1/2}\frac{S_{N-i}}{S_N}, \qquad i=0,\ldots,N,\ M\geqslant N. \end{aligned} \tag{101}$$

In general, comparing the method of polynomial algebras with the method of the Bethe ansatz one can conclude that according to the method of polynomial algebras eigenvectors (99) of the Hamiltonian are generated by an operator having the form of a polynomial while in the method of the Bethe ansatz the generating operator (19) has a factorized form. Correspondingly, in the first case we are interested in the coefficients Q_n of this polynomial, and in the second one in the roots λ_i of the related algebraic equation. In a special work we shall investigate this connection in more detail.

10. Conclusion

We have found that if $N > M$ the system of Bethe equations (23) can be weakened. In fact, for each M only a finite number of states are described by the usual system (23), but for an infinite number of states the reduced system (36) can be used. This system contains M equations for any $N > M$. With its help one can find their asymptotics for large N. The solutions of system (36) meet invariants (44). The other quantum optical models, such as the Dike model and four-boson interaction, have similar properties; we are going to investigate these subsequently. We also intend to investigate the connection between equations (23) and (36) and the polynomial algebras.

Acknowledgment

The research described in this publication is supported in part by grant no 02-02-17240 of the Russian Fundamental Research Foundation.

References

[1] Faddeev L D 1980 *Sov. Sci. Rev.* C **1** 107
[2] Kulish P P and Sklyanin E K 1981 *Lecture Notes in Physics* vol 151 p 61
[3] Bogoliubov P P, Izergin A G and Korepin V E 1985 *Lecture Notes in Physics* vol 242 p 220
Bogoliubov P P, Izergin A G and Korepin V E 1993 *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge: Cambridge University Press)
[4] De Vega H J 1989 *Int. J. Mod. Phys.* **4** 2371
[5] Shastry B S 1988 *J. Stat. Phys.* **50** 57
[6] Wadati M, Olmedilla E and Akutsu Y 1987 *J. Phys. Soc. Japan* **56** 1340
[7] Izergin A G and Korepin V E1981 *Lett. Math. Phys.* **5** 199
[8] Sklyanin E K 1989 *Nucl. Phys.* B **326** 719
[9] Kulish P P 1981 *Lett. Math. Phys.* **5** 191
[10] Gaudin M 1972 *Note CEA* 1559 p 1
Gaudin M 1973 *CEN-Saclay* 1559 p 2
[11] Sklyanin E K 1987 *Zap. Nauchn. Sem. LOMI* **164** 151

[12] Kuznetsov V B 1992 *J. Math. Phys.* **33** 3240
[13] Mcfarlane A J 1994 *Phys. Lett.* B **268** 71
[14] Brzezinski T and Mcfarlane A J 1994 *J. Math. Phys.* **35** 3261
[15] Scott D R D 1994 *J. Math. Phys.* **35** 5831
[16] Jurcŏ B 1990 *J. Math. Phys.* **30** 1289; 1739
[17] Rupasov V I 1982 *Zh. Eksp. Teor. Fiz.* **83** 1711
[18] Andreev V A and Ivanova O A 1992 *Phys. Lett.* A **171** 145
[19] Its A R, Izergin A G, Korepin V E and Slavnov N A 1990 *Int. J. Mod. Phys.* B **4** 1003
[20] Korepin V E and Slavnov N A 1991 *Commun. Math. Phys.* **136** 633
[21] Choudhury A G and Chowdhyry A R 1996 *Phys. Lett.* A **220** 205
[22] Andreev V A and Ivanova O A 1996 *Phys. Lett.* A **216** 76
[23] Karassiov V P 1994 *J. Phys. A: Math. Gen.* **27** 153
[24] Karassiov V P and Klimov V P 1994 *Phys. Lett.* A **189** 43
[25] Chumakov S M and Kozierowski M 1996 *Quantum Semiclass. Opt.* **8** 775
[26] Drobny G and Jex I 1993 *Opt. Commun.* **102** 141
[27] Drobny G, Bandilla A and Jex I 1997 *Phys. Rev.* A **55** 78